Ptolomey teoremasini analitik isbotlash

Ismatov Normurod[1]

[1]: Jizzax davlat pedagogika instituti

Matematika fanida, xususan, geometriya fanida biror masala yoki teoremani turli usullarda yechish yoki isbotlash muammolari juda muhim ahamiyatga ega. Jumladan, elementar geometriyaning ayrim muhim teoremalarini isbotlashda turli metodlarni qo‘llash va ularni amaliy tatbiqlari bayon qilish dolzarb masalalardan biridir. Shularni hisobga olib planimetriyaning ayrim, muhim teoremalarini tanlab olish ularning turli isbotlariga ahamiyat beramiz. Tanlangan teoremalarning elementar – sintetik usuldagi isbotlari bu sohadagi adabiyotlarda keng yoritilgan. Lekin, ularning analitik – koordinatalar usulidagi isbotlari ilmiy – uslubiy adabiyotlarda kam yaratilgan bo‘lib, bu geometriyani o‘rganishda muhim ahamiyatga ega.

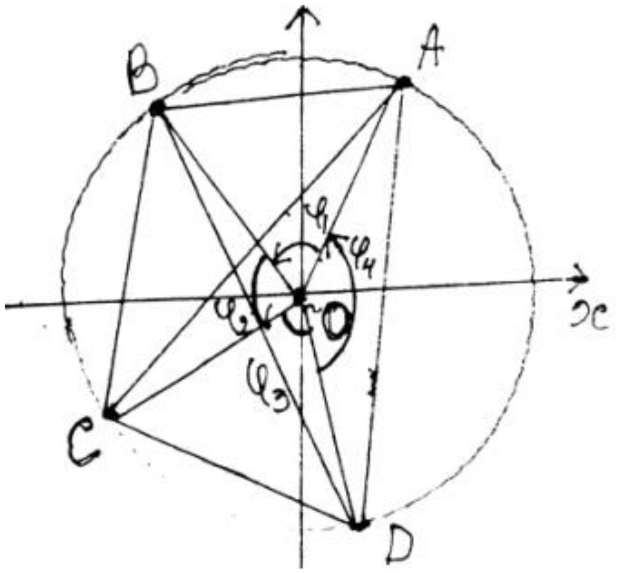
Quyida keltirilgan Ptolomey teoremasining elementar – sintetik usuldagi isbotlari o‘quv qo‘llanmalarda ko‘plab uchraydi. Lekin geometrik vektorlar yordamidagi isboti deyarli uchramaydi.

Teorema (Ptolomey). Doiraga ichki chizilgan to‘rtburchakning diagonallarining ko‘paytmasi to‘rtburchak qarama – qarshi tomonlari ko‘paytmalari yig‘indiiga teng.

Isbot. Markazi O koordinatalar boshida yotuvchi birlik alana olib unga ixtiyoriy $ABCD$ ichki to‘rtburchak chizamiz. (1-chizma). $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}, \overrightarrow{OD}$ vektorlari birlik vektorlardir. Quyidagi belgilashlarni kiritamiz.

$$\varphi_1 = (\overrightarrow{OA},\widehat{\ }\,\overrightarrow{OB}), \quad \varphi_2 = (\overrightarrow{OB},\widehat{\ }\,\overrightarrow{OC}), \quad \varphi_3 = (\overrightarrow{OC},\widehat{\ }\,\overrightarrow{OD})$$

$$\varphi_4 = (\overrightarrow{OD},\widehat{\ }\,\overrightarrow{OA}). \quad \varphi_1 + \varphi_2 + \varphi_3 + \varphi_4 = 2\pi .$$



1-chizma

Quyidagilarni yozishimiz mumkin:

$$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA},\quad AB^2=OB^2+OA^2-2\cdot\overrightarrow{OA}\cdot\overrightarrow{OB}=$$

$$=2-2|OB|\cdot|OA|\cdot\cos\varphi_1=2-2\cos\varphi_1;$$

$$\overrightarrow{BC}=\overrightarrow{OC}-\overrightarrow{OB};\quad BC^2=2-2\cos\varphi_2;$$

$$\overrightarrow{CD}=\overrightarrow{OD}-\overrightarrow{OC};\quad CD^2=2-2\cos\varphi_3;$$

$$\overrightarrow{DA}=\overrightarrow{OA}-\overrightarrow{OD};\quad DA^2=2-2\cos\varphi_4.$$

$$AB\cdot CD+BC\cdot AD=BD\cdot AC \tag{1}$$

tenglikni isbotlash talab etiladi.

$$AB\cdot CD=\sqrt{2-2\cos\varphi_1}\cdot\sqrt{2-2\cos\varphi_3}=2\sqrt{(1-\cos\varphi_1)(1-\cos\varphi_3)}$$

$1-\cos\alpha=2\sin^2\frac{\alpha}{2}$ formulaga asosan

$$AB\cdot CD=2\sqrt{2\sin^2\frac{\varphi_1}{2}\cdot 2\sin^2\frac{\varphi_3}{2}}=4\sin\frac{\varphi_1}{2}\cdot\sin\frac{\varphi_3}{2}$$

$$BC\cdot AD=\sqrt{2-2\cos\varphi_2}\cdot\sqrt{2-2\cos\varphi_4}=4\sin\frac{\varphi_2}{2}\cdot\sin\frac{\varphi_4}{2}.$$

$$\sin\alpha\cdot\sin\beta=\frac{1}{2}(\cos(\alpha-\beta)-\cos(\alpha+\beta))$$

ayniyatdan foydalanib

$$AB\cdot CD=2\left(\cos\frac{\varphi_1-\varphi_3}{2}-\cos\frac{\varphi_1+\varphi_3}{2}\right),\quad BC\cdot AD=2\left(\cos\frac{\varphi_2-\varphi_4}{2}-\cos\frac{\varphi_2+\varphi_4}{2}\right)$$

tengliklarni olish qiyin emas. Oxirgi tengliklarni hadma – had qo‘shib, quyidagilarni hosil qilamiz:

$$AB\cdot CD+BC\cdot AD=2\left[\cos\frac{\varphi_1-\varphi_3}{2}+\cos\frac{\varphi_2-\varphi_4}{2}-\left(\cos\frac{\varphi_1+\varphi_3}{2}+\cos\frac{\varphi_2+\varphi_4}{2}\right)\right].$$

Oxirgi tenglikka

$$\cos\alpha+\cos\beta=2\cos\frac{\alpha-\beta}{2}-\cos\frac{\alpha-\beta}{2}$$

formulani qo‘llaymiz. Natijada quyidagilarni yozishimiz mumkin:

$$\cos\frac{\varphi_1-\varphi_3}{2}+\cos\frac{\varphi_2-\varphi_4}{2}=2\cos\frac{\varphi_1+\varphi_2-(\varphi_3+\varphi_4)}{4}\cdot\cos\frac{\varphi_1+\varphi_4-(\varphi_2+\varphi_3)}{4}=$$

$$=2\cos\frac{2\pi-2(\varphi_3+\varphi_4)}{4}\cdot\cos\frac{2\pi-2(\varphi_2+\varphi_3)}{4}=2\cos\left(\frac{\pi}{2}-\frac{\varphi_2+\varphi_3}{2}\right)=$$

$$2\sin\frac{\varphi_3+\varphi_4}{2}\sin\frac{\varphi_2+\varphi_3}{2}.$$

$\varphi_1+\varphi_2+\varphi_3+\varphi_4=2\pi$ ekanligini hisobga olsak

$$\cos\frac{\varphi_1+\varphi_3}{2}+\cos\frac{\varphi_2+\varphi_4}{2}=2\cdot\cos\frac{\pi}{2}\cdot\cos\left(\frac{\pi}{2}-\frac{\varphi_2+\varphi_4}{2}\right)=0$$

ekanligini ko‘rish qiyin emas. Shunday qilib,

$$AB\cdot CD+BC\cdot AD=4\sin\frac{\varphi_3+\varphi_4}{2}\sin\frac{\varphi_2+\varphi_3}{2} \tag{2}$$

Endi $BD\cdot AC$ ko‘paytmaning qiymatini hisoblaymiz.

$$\overrightarrow{BD}=\overrightarrow{OD}-\overrightarrow{OB},\qquad BD^2=OD^2+OB^2-2\overrightarrow{OD}\cdot\overrightarrow{OB}=2-2\cos(\varphi_1+\varphi_4)$$

$$\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA},\qquad AC^2=2-2\cos(\varphi_1+\varphi_4)$$

$$BD\cdot AC=2\sqrt{(1-\cos(\varphi_1+\varphi_4))(1-\cos(\varphi_1+\varphi_2))}=4\sin\frac{\varphi_1+\varphi_4}{2}\sin\frac{\varphi_1+\varphi_2}{2}.$$

Ikkinchi tomondan

$$\sin\frac{\varphi_1+\varphi_4}{2}=\sin\frac{2\pi(\varphi_3+\varphi_2)}{2}=\sin\frac{\varphi_2+\varphi_3}{2};\quad \sin\frac{\varphi_1+\varphi_2}{2}=\sin\frac{\varphi_3+\varphi_4}{2}.$$

Demak,

$$BD\cdot AC=4\sin\frac{\varphi_2+\varphi_3}{2}\cdot\sin\frac{\varphi_3+\varphi_4}{2} \tag{3}$$

(2) va (3) tengliklardan (1) tenglikni hosil qilamiz.

Teorema to‘liq isbotlandi.

Foydalanilgan adabiyotlar ro‘yxati:

1. Dadajonov N.D., Jo‘rayeva M.SH. Geometriya. I qism. Toshkent, “O‘qituvchi”, 1996-224b.
2. Гусев В.А., Колягин Ю.М., Луканкин Г.Лю Векторы в школьном курсе математики. // Пособие для учителей.-М: Просвещение, 1976. – 47с
3. Nazarov N.N., Ochilova X.O., Podgoznova YE.G. Geometriyadan masalalar to‘plami. I qism. Toshkent, “O‘qituvchi”, 1997-87b.
4. Погорелев А.В. Геометрия: Учебник для 7-11 классов средней школы.- М.: Просвещение, 1992. – 383с.

Elektron ta’lim manbalari

1. www.ziyonet.uz
2. www.exponenta.ru
3. www.matlab.ru